

Vector-valued functions of multivariables

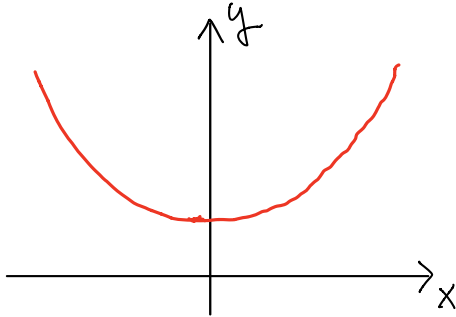
$\vec{F}: \Omega \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^m$. How to visualize it?

① Graph of \vec{F}

$$\text{Graph}(f) = \{ (\vec{x}, \vec{F}(\vec{x})) \in \mathbb{R}^{n+m} : \vec{x} \in \Omega \} \subseteq \mathbb{R}^{n+m}$$

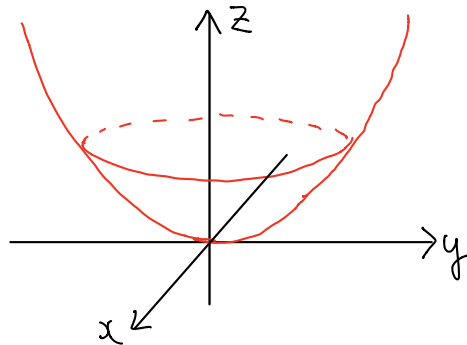
\uparrow \uparrow
 in \mathbb{R}^n in \mathbb{R}^m

eg $f: \mathbb{R} \rightarrow \mathbb{R}, f(x) = 1 + x^2$



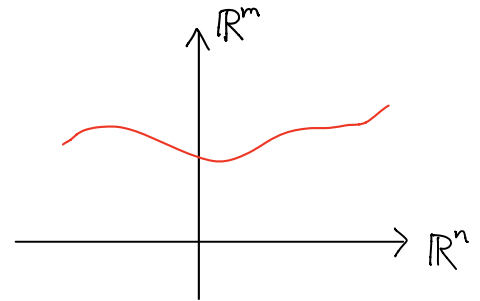
$\text{Graph}(f) \subseteq \mathbb{R} \times \mathbb{R} = \mathbb{R}^2$

$g: \mathbb{R}^2 \rightarrow \mathbb{R}, g(x,y) = x^2 + y^2$



$\text{Graph}(g) \subseteq \mathbb{R}^2 \times \mathbb{R} = \mathbb{R}^3$

In general $\vec{F}: \Omega \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^m$



Hard to draw if $n+m > 3$

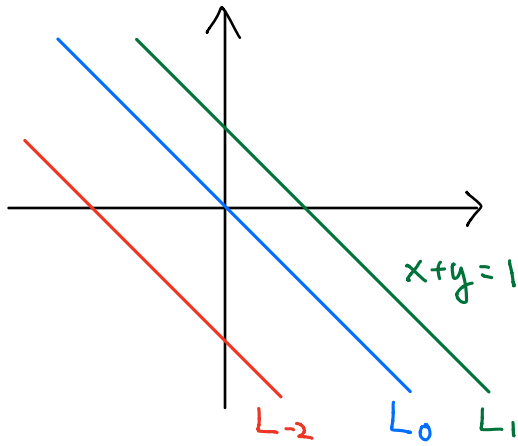
② Level set of $\vec{f}: \Omega \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^m$

If $c \in \mathbb{R}^m$, define the level set at c

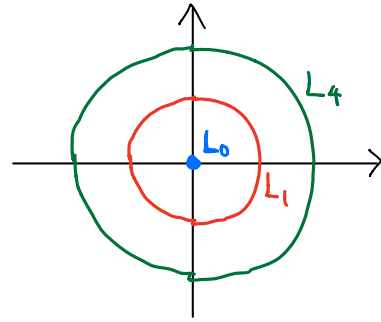
to be

$$L_c = \{x \in \Omega : \vec{f}(x) = \vec{c}\} = \vec{f}^{-1}(\vec{c}) \subseteq \Omega \subseteq \mathbb{R}^n$$

eg $f(x, y) = x + y \quad \Omega = \mathbb{R}^2$

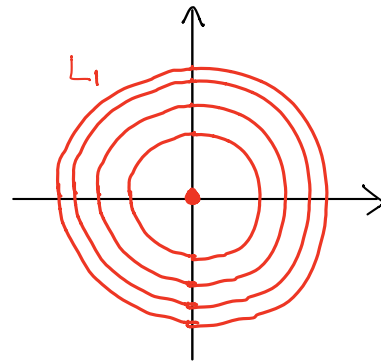


eg $g(x, y) = x^2 + y^2 \quad \Omega = \mathbb{R}^2$



L_c is $\begin{cases} \emptyset & \text{if } c < 0 \\ \text{a point} & \text{if } c = 0 \\ \text{a circle} & \text{if } c > 0 \end{cases}$

eg $h(x, y) = \cos(2\pi(x^2 + y^2)) \quad \Omega = \mathbb{R}^2$



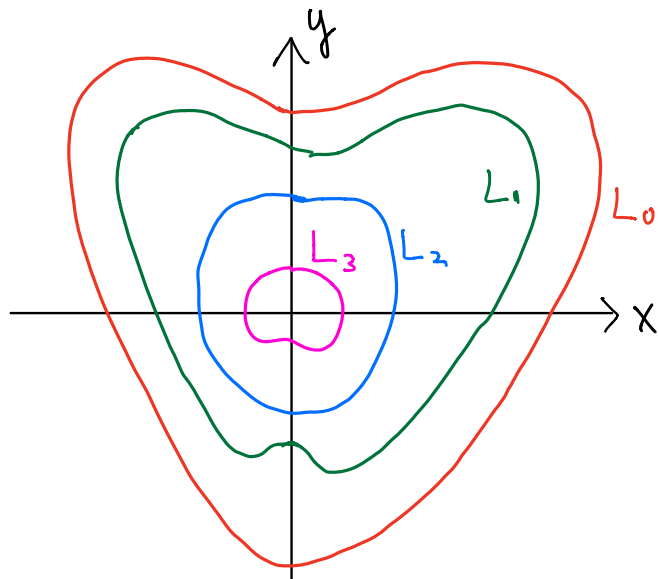
$$h(x, y) = 1 \Leftrightarrow x^2 + y^2 \in \mathbb{Z}$$

$\therefore L_1$ consists of infinitely many circles with radius \sqrt{k} , $k \geq 1$ and the origin

$$L_1 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \in \mathbb{Z}\}$$

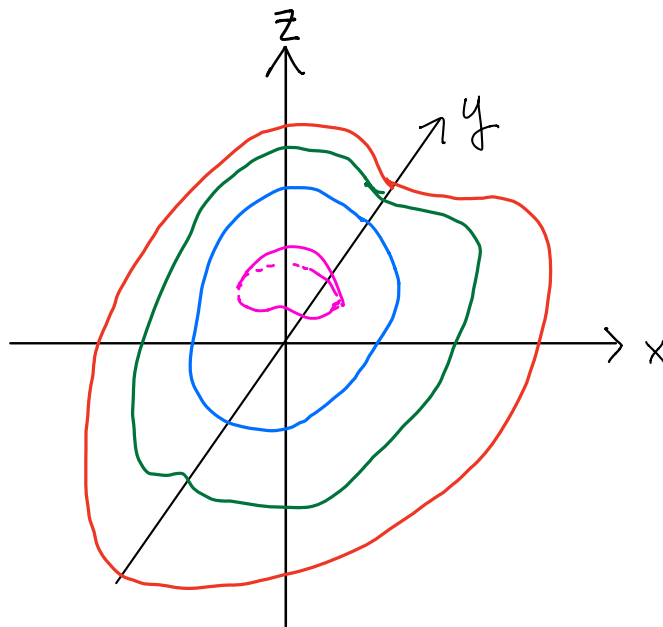
Relate level set and graph

eg $f: \mathbb{R}^2 \rightarrow \mathbb{R}$



Level sets of f
(drawn on domain = \mathbb{R}^2)

~ contour lines a map



Graph of f
(drawn on domain \times codomain = \mathbb{R}^3)

~ Mountain

Limit of multi-variable functions

Let $A \subseteq \mathbb{R}^n$. Define (An 2.5, 2.6)
(Thomas 14.2)

$$\bar{A} = A \cup \partial A = \text{closure of } A$$

For $a \in \bar{A}$ and $\vec{f}: A \rightarrow \mathbb{R}^m$

Want to define $\lim_{\vec{x} \rightarrow \vec{a}} f(\vec{x}) = \vec{L}$ to mean

If \vec{x} is very close to \vec{a} ,

$\vec{f}(\vec{x})$ is very close to \vec{L}

Close \leftrightarrow distance is small

Defn (ϵ - δ) $\vec{f}: A \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^m$.

We say that $\lim_{\vec{x} \rightarrow \vec{a}} \vec{f}(\vec{x}) = \vec{L}$ if

$\forall \epsilon > 0, \exists \delta > 0$ such that

if $\vec{x} \in A$ and $0 < \|\vec{x} - \vec{a}\| < \delta$

then $\|\vec{f}(\vec{x}) - \vec{L}\| < \epsilon$

Rmk ① $\forall =$ for all $\exists =$ there exists

② $\|\vec{x} - \vec{a}\| =$ distance between \vec{x} and \vec{a} in \mathbb{R}^n

$$0 < \|\vec{x} - \vec{a}\| < \delta$$

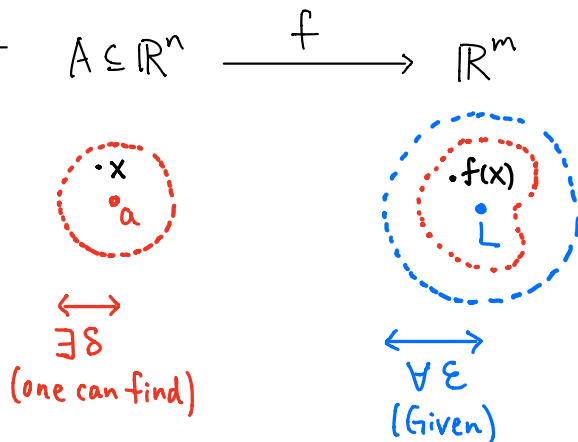
\uparrow
means $\vec{x} \neq \vec{a}$

\therefore Consider points close to \vec{a} but not equal to \vec{a}

③ $\|\vec{f}(\vec{x}) - \vec{L}\| =$ distance between $f(\vec{x})$ and \vec{L} in \mathbb{R}^m

If $m=1$, $\|\vec{f}(\vec{x}) - \vec{L}\| = |f(\vec{x}) - L| \leftarrow$ absolute value

Picture



eg $f: \mathbb{R}^2 \rightarrow \mathbb{R}$ $f(x,y) = x+y$

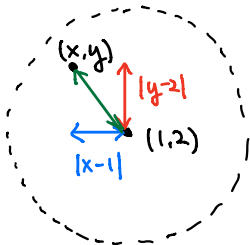
Illustrate that $\lim_{(x,y) \rightarrow (1,2)} f(x,y) = 3$

i.e.

Show that given any $\varepsilon > 0$,
one can find $\delta > 0$ such that
if $0 < \|(x,y) - (1,2)\| < \delta$,
then $|f(x,y) - 3| < \varepsilon$.

Idea: $|f(x,y) - 3| = |(x-1) + (y-2)|$
 $\leq |x-1| + |y-2|$

$$\|(x,y) - (1,2)\| = \sqrt{(x-1)^2 + (y-2)^2}$$



For example, for $\varepsilon = 1$, one can pick $\delta = \frac{1}{2}$:

If $\|(x,y) - (1,2)\| < \delta = \frac{1}{2}$, then

$$|x-1| = \sqrt{(x-1)^2} \leq \sqrt{(x-1)^2 + (y-2)^2} < \frac{1}{2}$$

$$|y-2| = \sqrt{(y-2)^2} \leq \sqrt{(x-1)^2 + (y-2)^2} < \frac{1}{2}$$

$$\Rightarrow |f(x,y) - 3| \leq |x-1| + |y-2| < \frac{1}{2} + \frac{1}{2} = 1 = \varepsilon \quad \checkmark$$

Similarly, for $\varepsilon = \frac{1}{100}$, one can pick $\delta = \frac{1}{200}$

In general, we need to do it for any $\varepsilon > 0$

For any given $\varepsilon > 0$, one can pick $\delta = \frac{\varepsilon}{2}$. Then

$$\|(x,y) - (1,2)\| < \delta = \frac{\varepsilon}{2}$$

$$\Rightarrow |f(x,y) - 3| = |x+y-3| \leq |x-1| + |y-2| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

$$\therefore \lim_{(x,y) \rightarrow (1,2)} f(x,y) = 3$$

eg Let $f(x,y) = x^2 + y^2$

Show that $\lim_{(x,y) \rightarrow (0,0)} f(x,y) = 0$ from definition

Sol Need to show that $\forall \epsilon > 0$,

$\exists \delta > 0$ such that

if $0 < \|(x,y) - (0,0)\| = \sqrt{x^2 + y^2} < \delta$

then $\|f(x,y) - 0\| = |x^2 + y^2| < \epsilon$

eg If $\epsilon = \frac{1}{100}$,

one can pick $\delta = \frac{1}{10}$ (or anything smaller)

In general, given $\epsilon > 0$, one can pick $\delta = \sqrt{\epsilon}$

Then for $0 < \|(x,y) - (0,0)\| < \delta$

$\|f(x,y) - (0,0)\| = |x^2 + y^2| = (\sqrt{x^2 + y^2})^2 < \delta^2 = \epsilon$

Let $A \subseteq \mathbb{R}^n$, $\vec{a} \in \bar{A}$, $\vec{f}: A \rightarrow \mathbb{R}^m$

$\vec{f}(\vec{x}) = \begin{bmatrix} f_1(\vec{x}) \\ f_2(\vec{x}) \\ \vdots \\ f_m(\vec{x}) \end{bmatrix}$ ← Each $f_i: A \rightarrow \mathbb{R}$ is called a component of \vec{f}

Prop

$\lim_{\vec{x} \rightarrow \vec{a}} \vec{f}(\vec{x}) = \vec{L} = \begin{bmatrix} l_1 \\ l_2 \\ \vdots \\ l_m \end{bmatrix} \iff \lim_{\vec{x} \rightarrow \vec{a}} f_i(\vec{x}) = l_i$
for $i = 1, 2, \dots, m$

Consequence:

It is good enough for us to focus on limit of real-valued functions $f: A \rightarrow \mathbb{R}$ ($m=1$)

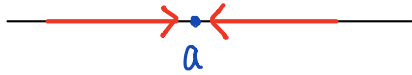
eg $\vec{f}: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ $\vec{f}(x,y) = \begin{bmatrix} x+y \\ x^2+y^2+1 \end{bmatrix}$
← $f_1(x,y)$
← $f_2(x,y)$

$\lim_{(x,y) \rightarrow (1,2)} \vec{f}(x,y) = \begin{bmatrix} \lim_{(x,y) \rightarrow (1,2)} x+y \\ \lim_{(x,y) \rightarrow (1,2)} x^2+y^2+1 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$

Limit along a path

In one variable:

Two ways to approach $a \in \mathbb{R}$

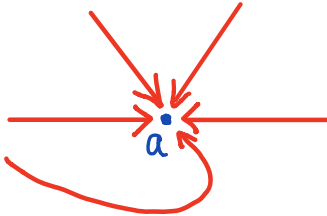


$$\lim_{x \rightarrow a} f(x) \text{ exists} \Leftrightarrow \lim_{x \rightarrow a^-} f(x) = \lim_{x \rightarrow a^+} f(x)$$

(exist and equal)

For n variables, $n \geq 2$

Many ways to approach $a \in \mathbb{R}^n$



The situation is not as simple
Need to consider all different curves to a

Fact $f: A \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$, $\bar{a} \in \bar{A}$

$$\lim_{\vec{x} \rightarrow \bar{a}} f(\vec{x}) = L \Leftrightarrow \text{limit of } f(\vec{x}) \text{ when } \vec{x} \text{ approaches to } \bar{a} \text{ along any path exists and equals to } L$$

Useful for showing limit does not exist (DNE)

- Find one path such that the limit along that path DNE

or

- Find two paths such that the limits along the two paths are different

$$\Rightarrow \lim_{\vec{x} \rightarrow \bar{a}} f(\vec{x}) \text{ DNE}$$

eg $\lim_{(x,y) \rightarrow (0,0)} \frac{x^2 - y^2}{x^2 + y^2}$ ← the function is not defined at (0,0)

Sol Look at limits along different paths

① Along x-axis ($y=0$)

$$\begin{aligned} \lim_{\substack{(x,y) \rightarrow (0,0) \\ y=0}} \frac{x^2 - y^2}{x^2 + y^2} &= \lim_{x \rightarrow 0} \frac{x^2 - 0^2}{x^2 + 0^2} \\ &= \lim_{x \rightarrow 0} 1 = 1 \end{aligned}$$

② Along y-axis ($x=0$)

$$\begin{aligned} \lim_{\substack{(x,y) \rightarrow (0,0) \\ x=0}} \frac{x^2 - y^2}{x^2 + y^2} &= \lim_{y \rightarrow 0} \frac{0^2 - y^2}{0^2 + y^2} \\ &= \lim_{y \rightarrow 0} -1 = -1 \neq 1 \end{aligned}$$

∴ Different limits along different paths

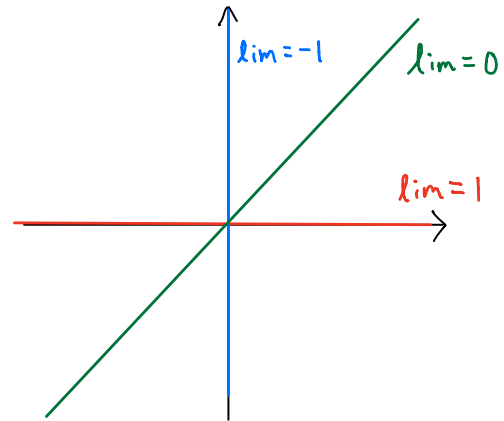
$$\Rightarrow \lim_{(x,y) \rightarrow (0,0)} \frac{x^2 - y^2}{x^2 + y^2} \text{ DNE}$$

Rmk We may look at other paths too

eg. Along $y=x$ (45°)

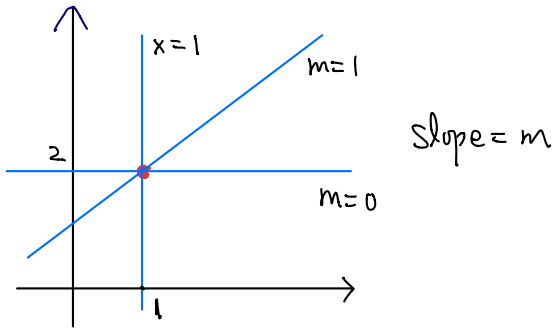
$$\begin{aligned} \lim_{\substack{(x,y) \rightarrow (0,0) \\ y=x}} \frac{x^2 - y^2}{x^2 + y^2} &= \lim_{x \rightarrow 0} \frac{x^2 - x^2}{x^2 + x^2} \\ &= \lim_{x \rightarrow 0} 0 = 0 \end{aligned}$$

What is limit along different slopes?



eg $\lim_{(x,y) \rightarrow (1,2)} \frac{xy - 2x - y + 2}{(x-1)^2 + (y-2)^2}$

Sol Find limit along different lines



① Along $x=1$

$$\begin{aligned} & \lim_{\substack{(x,y) \rightarrow (1,2) \\ x=1}} \frac{xy - 2x - y + 2}{(x-1)^2 + (y-2)^2} \\ &= \lim_{y \rightarrow 2} \frac{(1)y - 2(1) - y + 2}{(1-1)^2 + (y-2)^2} \\ &= \lim_{y \rightarrow 2} 0 = 0 \end{aligned}$$

② Along $y-2 = m(x-1)$

$$\begin{aligned} & \lim_{\substack{(x,y) \rightarrow (1,2) \\ y-2=m(x-1)}} \frac{xy - 2x - y + 2}{(x-1)^2 + (y-2)^2} \\ &= \lim_{\substack{(x,y) \rightarrow (1,2) \\ y-2=m(x-1)}} \frac{(x-1)(y-2)}{(x-1)^2 + (y-2)^2} \\ &= \lim_{x \rightarrow 1} \frac{m(x-1)^2}{(x-1)^2 + m^2(x-1)^2} \\ &= \frac{m}{1+m^2} \leftarrow \text{different limits for different } m \end{aligned}$$

eg. If $m=1$, limit = $\frac{1}{2}$
 If $m=0$, limit = 0

$\therefore \lim_{(x,y) \rightarrow (1,2)} \frac{xy - 2x - y + 2}{(x-1)^2 + (y-2)^2}$ DNE

eg $f: \mathbb{R}^2 \rightarrow \mathbb{R}$

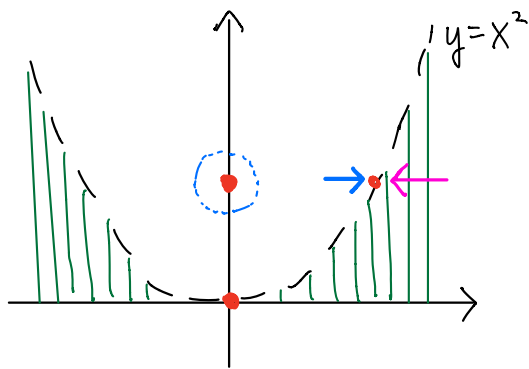
$$f(x,y) = \begin{cases} 1 & \text{if } 0 < y < x^2 \\ 0 & \text{otherwise} \end{cases}$$

Find $\lim_{(x,y) \rightarrow \vec{a}} f(x,y)$, where

i. $\vec{a} = (0,1)$

ii. $\vec{a} = (1,1)$

iii. $\vec{a} = (0,0)$



$f \equiv 1$ on  $f \equiv 0$ otherwise

Sol i $f \equiv 0$ near $(0,1) \Rightarrow \lim_{(x,y) \rightarrow (0,1)} f(x,y) = 0$

ii $\lim_{\substack{(x,y) \rightarrow (1,1) \\ y=1, x < 1}} f(x,y) = 0$ $\lim_{\substack{(x,y) \rightarrow (1,1) \\ y=1, x > 1}} f(x,y) = 1$
 different $\Rightarrow \lim_{(x,y) \rightarrow (1,1)} f(x,y)$ DNE

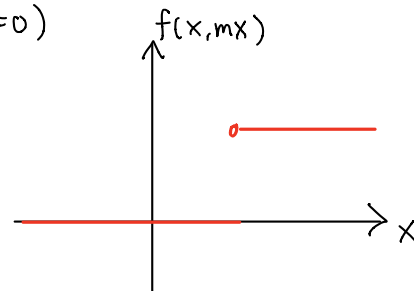
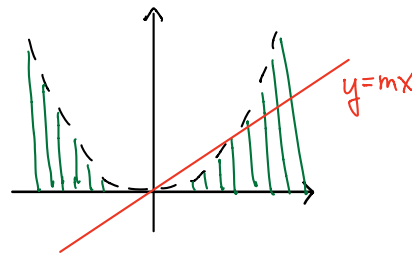
iii Case 1 Along y -axis ($x=0$)

$$f \equiv 0 \text{ on } y\text{-axis} \Rightarrow \lim_{\substack{(x,y) \rightarrow (0,0) \\ x=0}} f(x,y) = 0$$

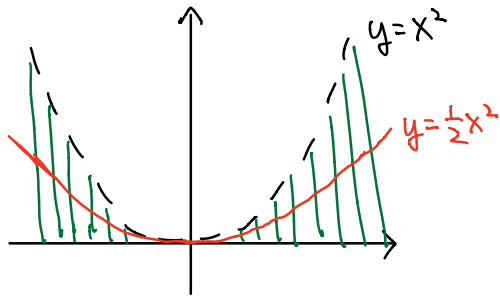
Case 2 Along $y=mx$

$$\lim_{\substack{(x,y) \rightarrow (0,0) \\ y=mx}} f(x,y) = \lim_{x \rightarrow 0} f(x, mx) = 0$$

If $m > 0$ (similar for $m < 0, m = 0$)



Case 3 Along the curve $y = \frac{1}{2}x^2$



$$\lim_{\substack{(x,y) \rightarrow (0,0) \\ y = \frac{1}{2}x^2}} f(x,y) = \lim_{x \rightarrow 0} f\left(x, \frac{1}{2}x^2\right)$$

$$= 1$$

$\neq 0$ as in case 1,2

$\therefore \lim_{(x,y) \rightarrow (0,0)} f(x,y)$ DNE

$$\circledast f\left(x, \frac{1}{2}x^2\right) = \begin{cases} 1 & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

Rmk Another way to show limit DNE is by ϵ - δ argument (Ex)

Properties of Limits

Assuming all limits on the right hand side exist then the limit on the left hand side exists and the formula holds

$$\textcircled{1} \lim_{\vec{x} \rightarrow \vec{a}} f(\vec{x}) \pm g(\vec{x}) = \lim_{\vec{x} \rightarrow \vec{a}} f(\vec{x}) \pm \lim_{\vec{x} \rightarrow \vec{a}} g(\vec{x})$$

$$\textcircled{2} \lim_{\vec{x} \rightarrow \vec{a}} k f(\vec{x}) = k \lim_{\vec{x} \rightarrow \vec{a}} f(\vec{x}) \quad \text{where } k \text{ is a constant.}$$

$$\textcircled{3} \lim_{\vec{x} \rightarrow \vec{a}} f(\vec{x}) g(\vec{x}) = \lim_{\vec{x} \rightarrow \vec{a}} f(\vec{x}) \cdot \lim_{\vec{x} \rightarrow \vec{a}} g(\vec{x})$$

$$\textcircled{4} \lim_{\vec{x} \rightarrow \vec{a}} \frac{f(\vec{x})}{g(\vec{x})} = \frac{\lim_{\vec{x} \rightarrow \vec{a}} f(\vec{x})}{\lim_{\vec{x} \rightarrow \vec{a}} g(\vec{x})} \quad \text{if } \lim_{\vec{x} \rightarrow \vec{a}} g(\vec{x}) \neq 0$$

$$\textcircled{5} \lim_{\vec{x} \rightarrow \vec{a}} [f(\vec{x})]^n = \left(\lim_{\vec{x} \rightarrow \vec{a}} f(\vec{x}) \right)^n, \quad n \geq 0$$

$$\textcircled{6} \lim_{\vec{x} \rightarrow \vec{a}} [f(\vec{x})]^{\frac{1}{n}} = \left[\lim_{\vec{x} \rightarrow \vec{a}} f(\vec{x}) \right]^{\frac{1}{n}} \quad \left(\begin{array}{l} \text{If } n \text{ is even,} \\ \text{assume } f(\vec{x}) \geq 0 \text{ near } \vec{a} \end{array} \right)$$

Squeeze theorem (Sandwich theorem)

Let $f, g, h: \Omega \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$

If $g(\vec{x}) \leq f(\vec{x}) \leq h(\vec{x})$ near $a \in \Omega$

and $\lim_{\vec{x} \rightarrow \vec{a}} g(\vec{x}) = \lim_{\vec{x} \rightarrow \vec{a}} h(\vec{x}) = L$

Then $\lim_{\vec{x} \rightarrow \vec{a}} f(\vec{x}) = L$

Rmk We say that a statement

$P(\vec{x})$ is true near $\vec{a} \in \mathbb{R}^n$ if

$P(\vec{x})$ is true $\forall \vec{x} \in B_\delta(\vec{a}) \setminus \{\vec{a}\}$ for some $\delta > 0$

Note $|f(\vec{x})| \leq g(\vec{x}) \Rightarrow -g(\vec{x}) \leq f(\vec{x}) \leq g(\vec{x})$

Hence,

Special case of Squeeze theorem

$|f(\vec{x})| \leq g(\vec{x})$ near \vec{a} and $\lim_{\vec{x} \rightarrow \vec{a}} g(\vec{x}) = 0$

$\Rightarrow \lim_{\vec{x} \rightarrow \vec{a}} f(\vec{x}) = 0$

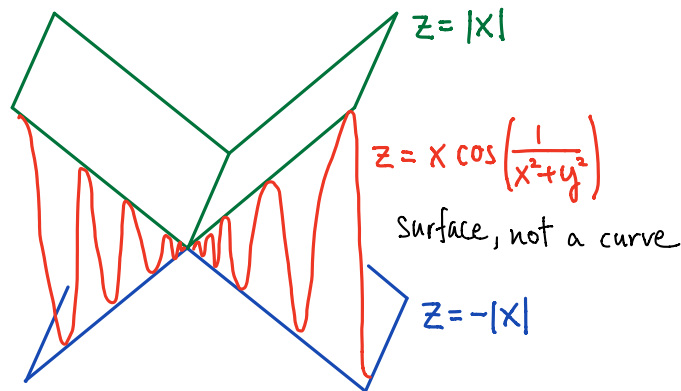
$$\text{eg } \lim_{(x,y) \rightarrow (0,0)} x \cos\left(\frac{1}{x^2+y^2}\right)$$

Sol Note

$$\left| \cos\left(\frac{1}{x^2+y^2}\right) \right| \leq 1 \Rightarrow \left| x \cos\left(\frac{1}{x^2+y^2}\right) \right| \leq |x|$$

Also, $\lim_{(x,y) \rightarrow (0,0)} |x| = 0$

$\therefore \lim_{(x,y) \rightarrow (0,0)} x \cos\left(\frac{1}{x^2+y^2}\right) = 0$ by squeeze thm



eg2 $\lim_{(x,y) \rightarrow (1,0)} \frac{(x-1)^2 \ln x}{(x-1)^2 + y^2}$

Sol Note

$$\left| \frac{(x-1)^2 \ln x}{(x-1)^2 + y^2} \right| = \left| \frac{(x-1)^2}{(x-1)^2 + y^2} \right| \cdot |\ln x|$$
$$\leq |\ln x|$$

Also, $\lim_{(x,y) \rightarrow (1,0)} |\ln x| = |\ln(1)| = 0$

By squeeze theorem,

$$\lim_{(x,y) \rightarrow (1,0)} \frac{(x-1)^2 \ln x}{(x-1)^2 + y^2} = 0$$

Rmk If $a \leq b$, then

$$ca \leq cb \quad \text{if } c > 0$$

$$ca \geq cb \quad \text{if } c < 0$$